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PROBLEM OF PLANE STRAIN OF HARDENING AND SOFTENING PLASTIC MATERIALS

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UDC 539.37

§1. **The classical description** of the kinematics of deformation of a solid medium is based on the assumption of sufficient smoothness of the displacement field. The smoothness assumption allows us to introduce the concept of a strain tensor and make use of the tool of differential equations for the description of the deformation of the material. However, there exist broad classes of motion of the medium where the displacement can be connected with the appearance of a plastic strain. Experiments with various materials show that the mechanism of plastic strain is connected with localization of shear along certain surfaces [1, 2]. The latter signifies that on these certain surfaces the displacement vector experiences a violent break. In the general case this circumstance turns out to be important and must be taken into account when describing plastic deformation. Making certain assumptions which are justified from a mechanical viewpoint, we can describe a nonsmooth displacement field by fairly simple means with the aid of a collection of smooth functions.

As an illustration, we shall consider the case of a single function of a single variable. By F(x) we denote the original function having a violent break at the points x_i. We assume that the distances between the breaks are small, that the function F(x) is sufficiently smooth between the break points, and that the values of the derivatives of F(x) on the right and on the left of the break points are equal to one another; i.e., the function

$$p(x)=\begin{cases} F'(x) & \text{for} \quad x\neq x_i, \\ F'(x_i\pm 0) & \text{for} \quad x=x_i \end{cases}$$

is sufficiently smooth.

Let f(x) be a smooth function satisfying the conditions $f(x_i)=F(x_i+0)$ and $P(x)=\int p(x)\,dx$. Then the original function F(x) can be characterized by a pair of smooth functions f(x), P(x) and a sequence of break points x_i (Fig. 1). The function f(x) has the meaning of averaging the original function, and it characterizes (with a certain accuracy) the values of F(x) over the entire domain of definition. The function $P'(x)-f'(x)$ characterizes the difference in local behavior of the original and the averaged functions, and for given break points determines the magnitude of jumps of the original function. Thus, a jump of the function F(x) at the point x_{i+1} with an accuracy up to l_i^2 equals $\{f'(x_i)-P'(x_i)\}l_i$ where $l_i=x_{i+1}-x_i$ is the distance between the adjacent break points.

Analogously, we shall consider the case of a vector function $\mathbf{V}=V_1\mathbf{e}_1+V_2\mathbf{e}_2$ of the vector argument $\mathbf{r}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$ ($\mathbf{e}_1$, $\mathbf{e}_2$ is the orthonormed basis). Let $l\ll 1$ be a characteristic dimension of the regions where the function $\mathbf{V}(\mathbf{r})$ is sufficiently smooth. We shall call such regions elements. We assume that for the original functions there exists a smooth average $\mathbf{v}(\mathbf{r})$ such that $\mathbf{v}(\mathbf{r}_i)=\mathbf{V}(\mathbf{r}_i)$ at the centers of elements $\mathbf{r}_i$; on the boundary separating elements with the centers at the points $\mathbf{r}_i$, $\mathbf{r}_{i+1}$ the break of $\mathbf{V}(\mathbf{r})$ with an accuracy up to $|\mathbf{r}_{i+1}-\mathbf{r}_i|^2$ equals $A(\mathbf{r}_i)(\mathbf{r}_{i+1}-\mathbf{r}_i)$, where A is a tensor of the second rank with smooth components A_{km}, k, m = 1, 2. A smooth vector field $\mathbf{v}(\mathbf{r})$ and a tensor field $A(\mathbf{r})$ are put in correspondence with the original field $\mathbf{V}(\mathbf{r})$. It can be shown that the description thus introduced imposes the following constraint on the class of discontinuous functions: On the break lines the values of one-sided derivatives $\partial V_k/\partial x_m$ must be equal to one another. It is obvious that

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 156–174, May–June, 1977. Original article submitted October 20, 1976.

$$A_{km} = \partial v_k/\partial x_m - \partial V_k/\partial x_m, \tag{1.1}$$

where the functions $\partial V_k/\partial x_m$ on the break lines are predetermined by their one-sided values.

Thus, certain classes of nonsmooth functions can be described by collections of smooth functions, one of which has the meaning of averaging the original function, while the rest characterize the breaks of the original function. As applied to the displacement field, such a description signifies that together with the smooth (averaged) field of displacements, new kinematic variables, which provide information about the breaks lost in the averaging process, are introduced.

The additional kinematic variables (on the basis of various hypotheses) have been introduced in many investigations (for example, in [3]); deformation of solids with a nonsmooth velocity field has directly been investigated in [4].

We consider an example of describing a discontinuous displacement field by means of smooth tensor and vector fields. We assume that the break lines are straight, orthogonal, and parallel to the coordinate axes. By x_1^i and x_2^j we denote the coordinates of intersection of the break lines with the corresponding axes. Let the smooth description of the field $\mathbf{V(r)}$ have the form

$$\begin{gathered} v_1 = sx_2,\ v_2 = 0, \\ A_{11} = 0,\ A_{22} = 0,\ A_{12} = s - s_e,\ A_{21} = 0, \end{gathered} \tag{1.2}$$

where the coefficients s and s_e depend only on the loading parameter. We consider the mechanical significance of the flow (1.2). First of all, from (1.2) it follows that the components of the displacement vector normal to the break lines are continuous: $A_{11} = \partial v_1/\partial x_1 - \partial V_1/\partial x_1 = 0$ and $A_{22} = \partial v_2/\partial x_2 - \partial V_2/\partial x_2 = 0$. The breaks of the shear components of the displacement vector on the lines $x_1 = x_1^i$ and $x_2 = x_2^j$ equal

$$\begin{gathered} \gamma_{12}^0 = A_{21} \frac{x_1^{i+1} - x_1^{i-1}}{2} = 0, \\ \gamma_{21}^0 = A_{12} \frac{x_2^{j+1} - x_2^{j-1}}{2} = (s - s_e) \frac{x_2^{j+1} - x_2^{j-1}}{2}. \end{gathered}$$

From (1.2) it also follows that $\partial V_1/\partial x_2 = -A_{12} + \partial v_1/\partial x_2 = s_e$ and $\partial V_2/\partial x_1 = -A_{21} + \partial v_2/\partial x_1 = 0$.

Thus, the description (1.2) and the data on x_1^i and x_2^j allow us to restore the discontinuous displacement field and the flow pattern: (1.2) represents a plane-parallel flow under the conditions where the breaks of the shear components of the displacement take place only along the family of lines $x_2 = \text{const}$ (Fig. 2a).

If

$$\begin{gathered} v_1 = sx_2,\ v_2 = 0, \\ A_{11} = 0,\ A_{22} = 0,\ A_{12} = 0,\ A_{21} = s - s_e, \end{gathered} \tag{1.3}$$

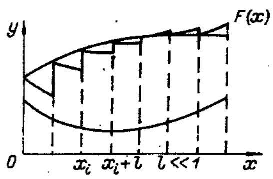
then

$$\gamma_{21}^0 = 0,\quad \gamma_{12}^0 = A_{21} \frac{x_1^{i+1} - x_1^{i-1}}{2} = (s - s_e) \frac{x_1^{i+1} - x_1^{i-1}}{2},$$

i.e., Eqs. (1.3) define a plane-parallel flow under the conditions where discontinuities of the tangential component of the displacement occur only along the lines $x_1 = \text{const}$ (Fig. 2b).

We shall call the break lines of the displacement vector slip lines, while the breaks of the tangent to the slip line of a component of the displacement vector will be called a slip or localized shear strain.

§2. We shall consider plane strain of a plastic medium. As was mentioned, plastic strains are connected



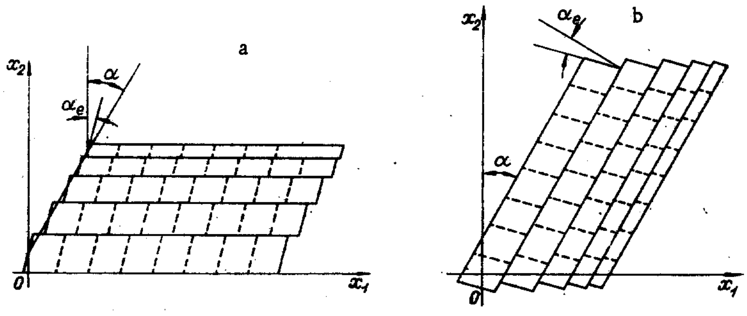


Fig. 1 Fig. 2

with localization of shear along certain lines (slip lines). It is natural to expect that this fact must be reflected in the corresponding plasticity equations. However, from the classical equations only the equations of ideal plasticity single out special directions (the directions of the characteristics of velocity and stress fields), which can be interpreted as a reflection of the shear mechanism of deformation along the slip lines. The equations of hardening plasticity are of the elliptic type, and they do not single out special directions in the development of the strain [5].

In [6] a method concerning the construction of models of media with an arbitrary rheological behavior, the mechanism of deformation of these media having a shear character along certain directions, was considered. Below we shall consider a realization of [6] for rigid-plastic and elastoplastic media with a diagram which can have rising and falling branches.

We assume that for certain external loads the region being deformed goes as a whole into a plastic state, the plasticity condition does not depend on the first invariant of the stress tensor, and the slip lines are orthogonal. By λ_1 and λ_2 we denote the parameters of the slip lines. Then

$$\frac{\partial x_2}{\partial \lambda_1} = \operatorname{tg}\left(\theta - \frac{\pi}{4}\right)\frac{\partial x_1}{\partial \lambda_1},\quad \frac{\partial x_2}{\partial \lambda_2} = \operatorname{tg}\left(\theta + \frac{\pi}{4}\right)\frac{\partial x_1}{\partial \lambda_2},$$
$$a_1 = \frac{\partial l_1}{\partial \lambda_1},\quad a_2 = \frac{\partial l_2}{\partial \lambda_2},\quad \frac{\partial a_1}{\partial \lambda_2} = -a_2\frac{\partial \theta}{\partial \lambda_1},\quad \frac{\partial a_2}{\partial \lambda_1} = a_1\frac{\partial \theta}{\partial \lambda_2},$$

where x_1 and x_2 are the Cartesian coordinates; $\theta - \pi/4$ is the angle between the tangent to the line λ_1 and the Ox_1 axis; and l_1 and l_2 are the lengths of arcs along the corresponding lines.

Let the deformation history of the material be known. Then the angle $\theta = \theta\,(x_1, x_2)$ is equal to the angle between the direction of the maximum principal stress and the Ox_1 axis at the instant when the maximum shear stresses at the point (x_1, x_2) reach a certain known magnitude k. To determine the angle θ and, consequently, the orientation of the grid of slip lines, we must solve the elastoplastic problem in the general case. In particular, if the material has a well-developed yield platform, then the orientation of the grid can be determined by methods of the theory of ideal plasticity [7]. In certain cases the orientation can be determined from symmetry conditions, diverse variational considerations, and so forth. We shall assume the angle θ to be known over the entire region of plastic deformation.

The second parameter characterizing the grid of slip lines is provided by the dimensions of the elements. We choose the two coordinate lines λ_2, λ_1 =const, which will be divided into elemental segments:

$$\Delta\lambda_1 = f_1(\lambda_1)\varepsilon,\quad \Delta\lambda_2 = f_2(\lambda_2)\varepsilon. \tag{2.1}$$

Equations (2.1) determine the partition of the entire plastic region into elements; the lengths of the sides of the element at the point (λ_1, λ_2) equal

$$l_1 = f_1(\lambda_1)a_1(\lambda_1, \lambda_2)\varepsilon,$$
$$l_2 = f_2(\lambda_2)a_2(\lambda_1, \lambda_2)\varepsilon. \tag{2.2}$$

Hence, in particular, it follows that the partition of the entire plastic region into correct elements is possible only for the condition $(\partial^2/\partial\lambda_1\partial\lambda_2)\ \ln\,(a_1/a_2) = 0$. In the general case such a partition is not possible.

Data on the physics of solids and experiments show that the plasticity limit of elements isolated by the grid of slip lines is much higher than the plasticity limit of the macrobody. If the macrobody is transformed into the plastic state for maximum shear stresses equal to k, then the elements are transformed into the plastic state for maximum shear stresses $k(l)$, where $k(l) \gg k$. We confine ourselves to such loading paths for which the shear stresses within the elements do not exceed $k(l)$. The larger the ratio $k(l)/k$, the broader the class of such loading paths.

The plasticity limit of an element depends on its dimension l. Therefore, we can assume that the stresses acting on the element determine its dimension; i.e., the denseness of the grid is determined by the stress field. The problems of fracturing of the elements, which can take place during hardening of the material, as well as development of slip lines from regions with the highest stress into regions with lower stresses, are not investigated in this work. We shall consider the deformation of the material after the formation of a grid of slip lines, where the functions $f_1\varepsilon$ and $f_2\varepsilon$, characterizing the denseness of the grid, are assumed to be known.

We assume that under shear no variation of the density (dilatation) of the material takes place and that the four components of the tensor A from (1.1) are determined only by two invariant functions Γ and Ω:

$$A_{11} = \cos 2\theta \cdot \Gamma,\ A_{22} = -\cos 2\theta \cdot \Gamma,$$
$$A_{21} = -\Omega + \sin 2\theta \cdot \Gamma,\ A_{12} = \Omega + \sin 2\theta \cdot \Gamma. \tag{2.3}$$

Conditions (2.3) signify that the components (normal to the sides of the elements) of the nonaveraged displacement field are continuous. The mechanical significance of the displacements follows from the equation

$$\Omega = \frac{1}{2}(A_{12} - A_{21}) = \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) - \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right),$$

i.e., the variable Ω characterizes the different rots (rot = curl) of the original and the averaged displacement fields. Together with the variable Ω, we shall use the variable ω:

$$\omega = \Omega + \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right),$$

which has the meaning of half the rot of the original nonsmooth displacement field.

In the example (1.2): $\frac{1}{2}$ rot $\mathbf{v} = -s/2$, $\Omega = (s - s_e)/2$, and $\omega = -s_e/2$. In the example (1.3): $\frac{1}{2}$ rot $\mathbf{v} = -s/2$, $\Omega = -(s - s_e)/2$, $\omega = -s + (s_e/2)$.

We shall show that the constraints on the class of discontinuous functions imposed by the description (1.1) are completely acceptable from a mechanical viewpoint. Indeed, on the break lines of displacements the shear and normal components of the stress tensor are continuous. We assume that all components are continuous. The derivatives $\partial V_k/\partial x_m$ determine the deformation and rotation of an element under the action of the stresses. Let the strain of the element be connected with the stresses by Hooke's law. Then continuity of the elastic strains follows from continuity of the stresses. (If on certain lines the stresses are discontinuous, then the analysis can be carried out by the usual methods.) Continuity of the rotations can be shown from the continuity conditions of the components of the displacement vector normal to the sides of the elements. Hence we have the continuity (equality of the one-sided values) of each of the derivatives $\partial V_k/\partial x_m = \partial v_k/\partial x_m - A_{km}$. It is obvious that for the components of the tensor of elastic strains $(1/2)(\partial V_k/\partial x_m + \partial V_m/\partial x_k)e_k e_m = (1/2)(\partial v_k/\partial x_m + \partial v_m/\partial x_k - A_{km} - A_{mk})e_k e_m$ the compatibility condition in the general case is not fulfilled and must not be fulfilled. Indeed, if from a smooth stress field we construct the field of elastic strains, then, in spite of the smoothness of the latter, there will be no compatibility, since the stresses in the general case do not satisfy the Beltrami–Mitchell equations. On the other hand, for the components of the tensor of total strains $(1/2)(\partial v_k/\partial x_m + \partial v_m/\partial x_k)e_k e_m$, the "total" displacements v_1 and v_2 (a smooth field of averaged displacements) do exist, and therefore the compatibility condition is fulfilled for the total strains.

The problem of "compatibility" connected with variability of the curvature of slip lines is more complicated. It will be shown below that the mechanism of deformation adopted coincides with the mechanism of deformation of an ideally plastic material, which can be introduced as an interpretation of an associated flow law (the Geiringer relations). Therefore, just as for an ideally plastic material, "compatibility" connected with variability of the curvature of slip lines will be fulfilled only for sufficiently small (localized) strains. We shall confine ourselves only to such strains.

§3. We shall consider the case where the elastic strains of the elements can be neglected. From the stiffness condition of the elements and the continuity condition of the components of the displacement vector normal to the sides of the elements we have the equations

$$\frac{\partial w_1}{\partial \lambda_1} - w_2 \frac{\partial \theta}{\partial \lambda_1} = 0, \quad \frac{\partial w_2}{\partial \lambda_2} + w_1 \frac{\partial \theta}{\partial \lambda_2} = 0, \tag{3.1}$$

where w_1 and w_2 are the projections of the displacement vector onto the normals to the slip lines λ_2 and λ_1.

We calculate the magnitude of the jump of the displacement component tangent to the side of the element. By V_n and v_n we denote the projections of the original and the averaged displacement vectors onto the n direction (Fig. 3). Then

$$\begin{aligned} \frac{\partial v_n}{\partial m} - \frac{\partial V_n}{\partial m} = \frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \\ + \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) - \frac{\cos 2\theta}{2}\left(\frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2}\right) - \frac{\sin 2\theta}{2}\left(\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\right) - \frac{1}{2}\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right). \end{aligned} \tag{3.2}$$

From (2.3), (3.2), and the stiffness condition of the element it follows that the magnitude of the jump is

$$\begin{aligned} \gamma_{12}^0 = \left(\frac{\partial v_n}{\partial m} - \frac{\partial V_n}{\partial m}\right) f_1 a_1 \varepsilon = (\Gamma - \Omega) f_1 a_1 \varepsilon = \\ = \left\{\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) - \Omega\right\} f_1 a_1 \varepsilon = \\ = \left(\frac{1}{a_1}\frac{\partial w_2}{\partial \lambda_1} + \frac{w_1}{a_1}\frac{\partial \theta}{\partial \lambda_1} - \omega\right) f_1 a_1 \varepsilon. \end{aligned}$$

Analogously, for the other side of the element

$$\begin{aligned} \gamma_{21}^0 = \left(\frac{\partial v_m}{\partial n} - \frac{\partial V_m}{\partial n}\right) f_2 a_2 \varepsilon = (\Gamma + \Omega) f_2 a_2 \varepsilon = \\ = \left\{\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \Omega\right\} f_2 a_2 \varepsilon = \\ = \left(\frac{1}{a_2}\frac{\partial w_1}{\partial \lambda_2} - \frac{w_2}{a_2}\frac{\partial \theta}{\partial \lambda_2} + \omega\right) f_2 a_2 \varepsilon. \end{aligned} \tag{3.4}$$

The values of the jumps explicity depend on the distances between the slip lines. The expressions (3.3) and (3.4) determine the localized shear strain (slip). In contrast to shear determined by the strain tensor and characterizing the variation of arbitrarily oriented angles, the quantities γ_{12}^0 and γ_{21}^0 have a meaning only on the corresponding slip lines and characterize actual shear (the slip having the dimension of length) of the elements along these planes.

The shear stresses which can develop on the slip planes are determined by the magnitude of localized strain on these planes, i.e.,

$$\sigma_{12}^0 = T(\gamma_{12}^0), \quad \sigma_{21}^0 = T(\gamma_{21}^0) \tag{3.5}$$

or

$$\gamma_{12}^0 = S(\sigma_{12}^0), \quad \gamma_{21}^0 = S(\sigma_{21}^0). \tag{3.6}$$

Here and in the following the index 0 marks variables referred to the corresponding slip planes.

In the formal role of the measure of shear on a slip plane we can take any quantity by which the values γ_{12}^0 and γ_{21}^0 can be restored (for example, the dimensionless quantity $\gamma_{12}^0/f_1 a_1 \varepsilon$, etc). The criterion of the choice of the measure is provided by relations of the type (3.5): For the measure of shear we must take a quantity which can be used in the defining equations. In all equations being considered, we have taken the breaks of displacements γ_{12}^0 and γ_{21}^0 which themselves have the dimensions of length as the measure of shear. If, however, for a certain material there occurs a possibility of using a dimensionless measure of plastic shear, then all the necessary calculations must be carried out analogously.

Equations (3.3) and (3.4) show that localized strains on slip planes of different families can be different even for $l_1 = l_2$. We write the condition of functioning of families of slip lines in the form

$$b_1 \gamma_{12}^0 + b_2 \gamma_{21}^0 = 0. \tag{3.7}$$

If one of the coefficients b_1 or b_2 is zero, then only one of the families of slip lines functions; if $b_1 + b_2 = 0$, then both families function identically, and so forth. The problem of choosing the coefficients will be considered below.

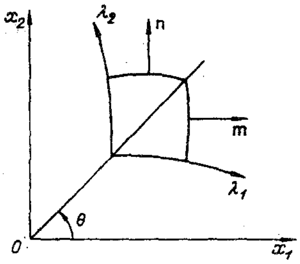


Fig. 3

Equations (3.1), (3.5), and (3.7), in combination with the equations of equilibrium, form a closed system relative to σ_{11}^0, σ_{22}^0, σ_{12}^0, σ_{21}^0, w_1, w_2, and ω:

$$\begin{gathered}
\frac{\partial}{\partial\lambda_1} a_2\sigma_{11}^0 + \frac{\partial}{\partial\lambda_2} a_1\sigma_{21}^0 - a_2\sigma_{12}^0\frac{\partial\theta}{\partial\lambda_1} - a_1\sigma_{22}^0\frac{\partial\theta}{\partial\lambda_2} + a_1a_2X_1^0 = 0,\\
\frac{\partial}{\partial\lambda_1} a_2\sigma_{12}^0 + \frac{\partial}{\partial\lambda_2} a_1\sigma_{22}^0 + a_2\sigma_{11}^0\frac{\partial\theta}{\partial\lambda_1} + a_1\sigma_{21}^0\frac{\partial\theta}{\partial\lambda_2} + a_1a_2X_2^0 = 0,\\
\frac{\partial w_1}{\partial\lambda_1} - w_2\frac{\partial\theta}{\partial\lambda_1} = 0, \quad \frac{\partial w_2}{\partial\lambda_2} + w_1\frac{\partial\theta}{\partial\lambda_2} = 0,\\
\sigma_{12}^0 = T\left[\left(\frac{1}{a_1}\frac{\partial w_2}{\partial\lambda_1} + \frac{w_1}{a_1}\frac{\partial\theta}{\partial\lambda_1} - \omega\right)a_1f_1\varepsilon\right],\\
\sigma_{21}^0 = T\left[\left(\frac{1}{a_2}\frac{\partial w_1}{\partial\lambda_2} - \frac{w_2}{a_2}\frac{\partial\theta}{\partial\lambda_2} + \omega\right)a_2f_2\varepsilon\right],\\
f_1b_1\frac{\partial w_2}{\partial\lambda_1} + f_2b_2\frac{\partial w_1}{\partial\lambda_2} + f_1b_1w_1\frac{\partial\theta}{\partial\lambda_1} - f_2b_2w_2\frac{\partial\theta}{\partial\lambda_2} + (b_2a_2f_2 - b_1a_1f_1)\,\omega = 0,
\end{gathered} \tag{3.8}$$

where X_1^0 and X_2^0 are the projections of the vector of body force onto the tangents to the slip lines λ_1 and λ_2.

The relations (3.5) show that cases in which $\sigma_{12}^0 \neq \sigma_{21}^0$ are possible. This signifies that for the maintenance of equilibrium on the sides of the elements distributed moments must make their appearance (body moments are assumed to be absent). From the equilibrium conditions we have the equation for the moments:

$$\frac{\partial}{\partial\lambda_1} a_2\mu_{11}^0 + \frac{\partial}{\partial\lambda_2} a_1\mu_{22}^0 + a_1a_2(\sigma_{12}^0 - \sigma_{21}^0) = 0. \tag{3.9}$$

The system (3.8), (3.9) is closed relative to all variables except μ_{11}^0 and μ_{22}^0. The indeterminacy of the moments is connected with the stiffness of the elements, and in each particular problem it is either eliminated by additional considerations or is retained. In the Cartesian coordinates the system (3.8), (3.9) is transformed to the form

$$\begin{gathered}
\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + X_1 = 0, \quad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + X_2 = 0,\\
\sin 2\theta\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) - \cos 2\theta\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) = 0,\\
-\sin 2\theta\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \cos 2\theta\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) = 0,\\
\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{2}\sin 2\theta - \frac{\sigma_{21}-\sigma_{12}}{2} = T\left(\left[\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) - \Omega\right]a_1f_1\varepsilon\right),\\
\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{2}\sin 2\theta + \frac{\sigma_{21}-\sigma_{12}}{2} = T\left(\left[\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \Omega\right]a_2f_2\varepsilon\right),\\
\cos 2\theta\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \sin 2\theta\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + 2\Lambda\Omega = 0,\\
\frac{\partial\mu_{11}}{\partial x_1} + \frac{\partial\mu_{22}}{\partial x_2} + (\sigma_{12} - \sigma_{21}) = 0,
\end{gathered} \tag{3.10}$$

where we have introduced the usual notation and $\Lambda = (b_2a_2f_2 - b_1f_1a_1)/(b_2a_2f_2 + b_1f_1a_1)$. System (3.10) is of the hyperbolic type. Slight discontinuities of the displacements v_1 and v_2 are possible only on the slip lines λ_1 and λ_2. At the same time, if a weak discontinuity is realized, then in the general case it gives rise to an intense discontinuity of the stresses. From the equations of equilibrium and (3.5) it follows that intense discontinuities of the stresses under conditions where the field of displacements v_1 and v_2 is smooth are possible only on the slip lines λ_1 and λ_2.

The third and fourth equations of system (3.10) show that the plastic potential

$$\Phi(\sigma_{km},\theta)=\frac{\sigma_{12}^0+\sigma_{21}^0}{2}=\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta+\frac{\sigma_{12}+\sigma_{21}}{2}\sin 2\theta,$$

$$\varepsilon_{11}=\frac{\partial v_1}{\partial x_1}=\lambda\frac{\partial\Phi}{\partial\sigma_{11}},\quad \varepsilon_{22}=\frac{\partial v_2}{\partial x_2}=\lambda\frac{\partial\Phi}{\partial\sigma_{22}}, \tag{3.11}$$

$$\varepsilon_{12}=\varepsilon_{21}=\frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)=\lambda\frac{\partial\Phi}{\partial\sigma_{12}}=\lambda\frac{\partial\Phi}{\partial\sigma_{21}}$$

exists for the strains. Since for the loading paths considered the angle θ depends only on the coordinates, (3.11) can easily be transformed into relations for the strain rates.

The last equations of the system (3.11) can be regarded as relations determining the loading surface:

$$|\varphi_1(\sigma_{km},\theta)|-|T[(\lambda/2-\Omega)a_1f_1\varepsilon]|=\Phi_1(\sigma_{km},\varepsilon_{km},x_1,x_2,\theta)=0,$$
$$|\varphi_2(\sigma_{km},\theta)|-|T[(\lambda/2+\Omega)a_2f_2\varepsilon]|=\Phi_2(\sigma_{km},\varepsilon_{km},x_1,x_2,\theta)=0, \tag{3.12}$$
$$\varphi_1=\sigma_{12}^0(\sigma_{km},\theta);\quad \varphi_2=\sigma_{21}^0(\sigma_{km},\theta);$$

where

$$\lambda=\cos 2\theta(\varepsilon_{11}-\varepsilon_{22})+2\sin 2\theta\varepsilon_{12}=\frac{\gamma_{12}^0}{f_1a_1\varepsilon}+\frac{\gamma_{21}^0}{f_2a_2\varepsilon};\quad \Omega=-\frac{\lambda}{2A}.$$

The loading surface in the general case is singular and consists of the two surfaces Φ_1 and Φ_2: The region $\Phi_1<0$, $\Phi_2<0$ corresponds to the rigid state of the material, while the region Φ_1, $\Phi_2=0$ corresponds to the plastic state. The form of the expression (3.12) is chosen with account taken of the fact that the function (functional) T is odd.

The concepts of loading and unloading are defined in the space of strains: Loading takes place if at least one of the quantities γ_{12}^0 or γ_{21}^0 varies. If both quantities are fixed, then unloading or neutral loading takes place: unloading if $\frac{\partial|\varphi_{1,2}|}{\partial\sigma_{km}}d\sigma_{km}<0$, and neutral loading if $\frac{\partial|\varphi_i|}{\partial\sigma_{km}}d\sigma_{km}=0$, where $i=1$ or 2.

Thus, Eqs. (3.10) show the following: In the general case, as a result of plastic deformation, the material becomes anisotropic and inhomogeneous; the form of anisotropy is connected with the orientation of the grid of slip lines, while inhomogeneity is connected with the denseness of this grid; a potential exists for the tensor of plastic strain; the loading surface is singular; and the deformation law is nonassociative. In particular cases, we can have situations where the loading surface is smooth, the deformation law is associative, and the material is homogeneous. If $T'\equiv 0$, then Eqs. (3.10) are transformed into the equations of ideal plasticity. The model of ideal plasticity, which can be considered as a limit case of (3.10), possesses a number of exceptional properties: In spite of the possible inhomogeneity of the material, connected with the geometry and denseness of the grid of slip lines, the deformation equations do not depend on this inhomogeneity, nor do they depend on the possible difference in the functioning of slip lines belonging to different families; the stress tensor is symmetric even in the case of unequal localized shear strains (slippage) on planes belonging to different families.

We shall consider the problem of indentation of a smooth rigid die into a weightless half-space of plastic material with a yield platform. We assume that in the case of an ideal flow a plastic Hill zone develops in the material (Fig. 4): $\theta\equiv\pi$ in the region $A_1A_2A_3$, $\theta=\beta+\pi/4$ in the region $A_2A_3A_5$, and $\theta\equiv\pi/2$ in the region $A_3A_4A_5$, where r and β are polar coordinates with a pole at the point A_3 (the flow is symmetrical about the straight line $x_1=-a$).

We shall construct the solution of the system (3.10) for these values of θ and the following boundary conditions: On the boundary with the rigid region the normal displacement component is continuous, on the boundary A_1A_3 the shear stresses are absent and the vertical displacement is h, the boundary A_3A_4 is free from stresses, and on the lines A_2A_3 and A_3A_5 the corresponding displacements, stresses, and moments are continuous. We introduce the curvilinear coordinates λ_1 and λ_2 (see Fig. 4). It can be shown that $\sigma_{21}^0=T(0)=k$ in the region $A_1A_2A_3$ while $\sigma_{\beta r}=T\left(\frac{b_1f_1}{b_2f_2r-b_1f_1}\frac{\sqrt{2}h}{r}\right)$ in the region $A_2A_3A_5$. From the condition of continuity of the stresses on A_3A_2 it follows that

$$T\left(\frac{-b_1f_1}{b_2f_2r-b_1f_1}\frac{\sqrt{2}h}{r}\right)=T(0)=k. \qquad (3.13)$$

The last equation allows us to solve the problem of functioning of slip lines belonging to different families [i.e., to determine the coefficients b_1 and b_2 in Eq. (3.7)]. If the flow is ideal $[T(\gamma_{21}^0)\equiv k]$, then condition (3.13) does not impose any constraints on b_1 and b_2. If $T(\gamma_{21}^0)\neq\text{const}$, then from (3.13) it follows that $b_1=0$. Consequently, the continuity condition (3.13) shows that in the case of an ideal flow (hardening or softening) localized shear is possible only along the family of lines λ_2. Thus, in the given case the loading conditions uniquely determine the mode of deformation.

A solution satisfying the boundary conditions enumerated above has the following form:

in the region $A_3A_5A_4$:

$$v_1=h,\ v_2=-h,\ \omega=0,$$
$$\sigma_{11}=-2k,\ \sigma_{22}=\sigma_{12}=\sigma_{21}=0,\ \mu_{11}=\mu_{22}=0; \qquad (3.14)$$

in the region $A_3A_4A_5$: $v_r=0$, $v_\beta=\sqrt{2}h$, $\omega=-\sqrt{2}h/r$,

$$\sigma_{r\beta}=T\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right),\quad \sigma_{\beta r}=k,\quad \sigma_{rr}=-\left(\beta-\frac{\pi}{4}\right)\left[T\left(\frac{\sqrt{2}h}{2}f_1\varepsilon\right)+k\right]-k,$$
$$\sigma_{\beta\beta}=-\left[\frac{\partial}{\partial r}rT\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right)+k\right]\left(\beta-\frac{\pi}{4}\right)-k,$$
$$\mu_{rr}=-\frac{1}{r}\int_0^r\left[T\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right)-k\right]rdr,\quad \mu_{\beta\beta}=0;$$

in the region $A_1A_2A_3$: $v_1=h$, $v_2=h$, $\omega=0$, $\mu_{11}=0$, $\mu_{22}=0$,

$$\sigma_{11}=k+\frac{1}{2}\left[\Psi\left(\frac{-x_1-x_2+a}{\sqrt{2}}\right)+\Psi\left(\frac{-x_1+x_2+a}{\sqrt{2}}\right)\right],$$
$$\sigma_{22}=-k+\frac{1}{2}\left[\Psi\left(\frac{-x_1-x_2+a}{\sqrt{2}}\right)+\Psi\left(\frac{-x_1+x_2+a}{\sqrt{2}}\right)\right],$$
$$\sigma_{12}=\sigma_{21}=-\frac{1}{2}\left[\Psi\left(\frac{-x_1-x_2+a}{\sqrt{2}}\right)-\Psi\left(\frac{-x_1+x_2+a}{\sqrt{2}}\right)\right],$$

where

$$\Psi(r)=-\frac{\pi}{2}\left\{\frac{\partial}{\partial r}\left[rT\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right)\right]+k\right\}-k.$$

When integrating the equations it was additionally assumed that σ_{rr} and μ_{rr} have no singularity at the point $r=0$ and $\partial\mu_{\beta\beta}/\partial\beta=0$. The indentation condition and the diagram of stresses under the die, dependent on the depth h, are given by the expression

$$\sigma_{22}|_{x_2=0}=-k+\Psi((-x_1+a)/\sqrt{2}). \qquad (3.15)$$

In the particular case of an ideally plastic material the solution and the limit load (3.15) are transformed into the classic equivalents.

In the formulation being considered the problem is posed as a problem of initial plastic flow, i.e., h in the solution (3.14) must be small. Since in a centralized wave $a_2=r$, then as $r\to 0$ the dimensions of the elements along λ_2 become vanishingly small. We assume that along λ_1 the dimensions are also small, i.e., $f_1\varepsilon\sim r$ if $r\to 0$. Then the localized shear at the point $r=0$ has no singularity, and for a sufficiently small h the flow of the entire material will be ideal. By γ_* we denote the value of shear for which hardening or softening begins. Let the ratio γ_*/a be so small that the flow for h on the order γ_* can be assumed to be a starting flow. For these values of h yet another circumstance connected with the special state of the layer of material adjoining the line separating the rigid and plastic regions arises.

From the solution it follows that on the line $A_1A_2A_5A_4$ the shear component of the displacement vector is discontinuous and the break is equal to $\sqrt{2}h$. In connection with the fact that we have taken quantities having the dimension of length as the measure of localized shear in (3.5), the "shear" along the line $A_1A_2A_5A_4$, in principle, does not differ in any respect from the localized shear within the region of deformation. Therefore, on the boundary with the rigid region there emerges an additional boundary condition on the stresses:

$$\sigma_{12}^0=T(\sqrt{2}h). \qquad (3.16)$$

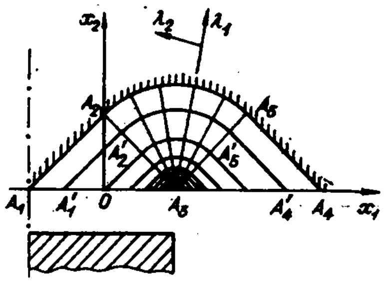


Fig. 4

Since the distances between slip lines are finite, the additional condition (3.16) does not contradict the solution constructed. Indeed, the stresses (3.14) have significance only up to the line $A'_1A'_2A'_5A'_4$. Therefore, the "boundary layer" $A_1A_2A_5A_4A'_4A'_5A'_2A'_1$, on the one hand, is under the effect of the shear stresses (3.14), while, on the other hand, it is under the effect of the shear stresses (3.16). This leads to the fact that the stress σ_{22}^0 is discontinuous on the line $A'_1A'_2A'_5A'_4$, while in the "boundary" layer it will vary rapidly along λ_2. From the conditions of equilibrium of the elements forming the layer $A_1A_2A_5A_4A'_4A'_5A'_2A'_1$ we find

$$\sigma_{22}|_{A_1A_1'} = -\frac{\pi}{2}\left[T\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right)+k-\frac{T(\sqrt{2}h)-T\left(\frac{\sqrt{2}h}{r}f_1\varepsilon\right)}{A_1A_1'}a\sqrt{2}\right]-2k-\frac{2a\sqrt{2}}{A_1A_1'}[T(\sqrt{2}h)-k].$$

We note that $T(0)\leq k$, and not equal to k, as was assumed above. If $T(0)<k$, then unloading can take place in regions where $\gamma_{12}^0=0$, $\gamma_2^0=0$. This variant can be investigated analogously to that considered above.

§4. We proceed to the model of an elastoplastic body. We assume that the deformation of the elements is completely reversible, while all localized deformation (slip) is irreversible. We shall first consider the problem of elastic strains of elements under the stresses σ_{km}^0. The normal stresses σ_{kk}^0 give rise to an extension of the element in the direction of its action and a compression in the lateral direction. From the condition of continuity of the components of the displacement vector normal to the sides of the element it follows that the compression strains coincide for the original smooth and the averaged smooth displacement fields. Consequently,

$$\begin{gathered}\frac{1}{a_1}\frac{\partial w_1}{\partial\lambda_1}-\frac{w_2}{a_1}\frac{\partial\theta}{\partial\lambda_1}=\frac{1-\nu}{2\mu}\sigma_{11}^0-\frac{\nu}{2\mu}\sigma_{22}^0,\\ \frac{1}{a_2}\frac{\partial w_2}{\partial\lambda_2}+\frac{w_1}{a_2}\frac{\partial\theta}{\partial\lambda_2}=\frac{1-\nu}{2\mu}\sigma_{22}^0-\frac{\nu}{2\mu}\sigma_{11}^0,\end{gathered} \tag{4.1}$$

where μ is the shear modulus and ν is Poisson's ratio. Just as for the rigid-plastic body, we can show that the shear stresses σ_{12}^0 and σ_{21}^0 in the general case are not equal to one another. The action of the stresses σ_{12}^0 and σ_{21}^0 is represented as a superposition of the two systems of stresses τ_+^0 and τ_-^0: $\sigma_{12}^0=\tau_+^0+\tau_-^0$ and $\sigma_{21}^0=\tau_+^0-\tau_-^0$. From the symmetry considerations it follows that the elastic shear of the element takes place only under the effect of the component $\tau_+^0=(\sigma_{12}^0+\sigma_{21}^0)/2$. The value of the elastic shear is

$$\frac{\cos 2\theta}{2}\left(\frac{\partial V_1}{\partial x_1}-\frac{\partial V_2}{\partial x_2}\right)+\frac{\sin 2\theta}{2}\left(\frac{\partial V_2}{\partial x_1}+\frac{\partial V_1}{\partial x_2}\right)=\frac{\sigma_{12}^0+\sigma_{21}^0}{4\mu}. \tag{4.2}$$

We consider the equations for localized strain. By definition, localized shear on a slip plane equals

$$\begin{aligned}\gamma_{12}^0=\left(\frac{\partial v_n}{\partial m}-\frac{\partial V_n}{\partial m}\right)f_1a_1\varepsilon=\Bigg\{&\left[\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)+\frac{\sin 2\theta}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)-\Omega\right]-\\ &-\left[\frac{\cos 2\theta}{2}\left(\frac{\partial V_1}{\partial x_1}-\frac{\partial V_2}{\partial x_2}\right)+\frac{\sin 2\theta}{2}\left(\frac{\partial V_1}{\partial x_2}+\frac{\partial V_2}{\partial x_1}\right)\right]\Bigg\}f_1a_1\varepsilon.\end{aligned} \tag{4.3}$$

The relations (3.6), reflecting the connection between localized shear and the corresponding shear stress, remain unaltered. Substituting (4.2) into (4.3) and then into (3.6), we obtain the equation

$$\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)+\frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)-\Omega=\frac{1}{f_1a_1\varepsilon}S(\sigma_{12}^0)+\frac{\sigma_{12}^0+\sigma_{21}^0}{4\mu}. \tag{4.4}$$

Analogously, for the other slip plane

$$\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)+\frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right)+\Omega=\frac{1}{f_2 a_2 \varepsilon} S\left(\sigma_{21}^0\right)+\frac{\sigma_{21}^0+\sigma_{12}^0}{4\mu}. \tag{4.5}$$

In an elastoplastic body the elements deform both under the effect of the stresses σ_{km}^0 and under the effect of moments. The last circumstance allows us to close the system relative to all variables, including the moments.

From the above mechanism of deformation it follows that the forces between the elements are transmitted via the stresses distributed over their sides. Therefore, the moments μ_{11}^0 and μ_{22}^0 can arise only as a result of nonuniformity of such a distribution. In the given case the stress distribution over the side of the element can acceptably be assumed to be linear. The linear diagram is represented as a superposition of two diagrams: a constant one, which gives the same force along the side of the element as the original diagram but does not give any moment, and a linear one, which as a sum does not give any force but does give a moment. We denote the maximum stress on the side λ_2 of the element corresponding to the second diagram by Σ_{11}^0. Then

$$\mu_{11}^0=\Sigma_{11}^0 a_2 f_2 \varepsilon / 6.$$

Analogously, for the side λ_1

$$\mu_{22}^0=\frac{\Sigma_{22}^0}{6} a_1 f_1 \varepsilon.$$

The nonuniform stresses acting on the side λ_2 of the element cause not only an extension of the element, which is taken into account by the first equation of (4.1), but also a rotation of this side through an angle Ω_{11}^0. By means of the usual relation we can connect the variation of Ω_{11}^0 along λ_1 (Ω_{11}^0 is smooth along λ_1) with μ_{11}^0:

$$\frac{1}{a_1} \frac{\partial \Omega_{11}^0}{\partial \lambda_1}=\frac{12 \mu_{11}^0}{\mathrm{E}\left(a_2 f_2 \varepsilon\right)^2}, \tag{4.6}$$

where E is Young's modulus.

Analogously, along λ_2

$$\frac{1}{a_2} \frac{\partial \Omega_{22}^0}{\partial \lambda_2}=\frac{12 \mu_{22}^0}{\mathrm{E}\left(a_1 f_1 \varepsilon\right)^2}. \tag{4.7}$$

If we follow the ideas of the moment theory of elasticity, then we have to put $\Omega_{11}^0=\Omega_{22}^0$ [8] (in the given case it is unimportant whether the variable $\Omega_{11}^0=\Omega_{22}^0$ is independent or is determined, as in [8], by the rot of the displacement field) and obtain the closing equation for the moments by cross differentiation of (4.6) and (4.7). However, the mechanism of deformation adopted above leads to the necessity of additionally introducing three kinematic variables at the point λ_1, λ_2: Ω, Ω_{11}^0, and Ω_{22}^0. At the same time, a priori we have no justification for the assumption that the variables Ω_{11}^0 and Ω_{22}^0 are connected with one another and with the variable Ω. Therefore, the two equations (4.6) and (4.7) introduce the two new unknowns Ω_{11}^0 and Ω_{22}^0 into the system of equations and do not allow us to directly close the system relative to the moments.

The closing equations can be obtained in the following manner. By W we denote the part of the elastic energy which is stored in the region D of the elastoplastic body as result of deformation of elements under effect of the moments μ_{11}^0 and μ_{22}^0. It can be shown that

$$W=\frac{6}{\mathrm{E}} \int_D\left[\frac{\left(\mu_{11}^0\right)^2}{\left(a_2 f_2 \varepsilon\right)^2}+\frac{\left(\mu_{22}^0\right)^2}{\left(a_1 f_1 \varepsilon\right)^2}\right] a_1 a_2 d \lambda_1 d \lambda_2. \tag{4.8}$$

We assume that out of all possible moment distributions satisfying the equations of equilibrium and boundary conditions the one that makes the potential energy W a minimum is realized. The minimum of the functional (4.8) under the condition (3.9) is realized if the moments satisfy the equation

$$\frac{\partial}{\partial \lambda_1} \frac{a_2 \mu_{22}^0}{\left(a_1 f_1 \varepsilon\right)^2}=\frac{\partial}{\partial \lambda_2} \frac{a_1 \mu_{11}^0}{\left(a_2 f_2 \varepsilon\right)^2}. \tag{4.9}$$

In all cases considered above the functions l_1 and l_2 from (2.2) can be interpreted as nonhomogeneous characteristics of the material having the dimension of length. If the grid of slip lines is regular and consists of straight lines ($l_1 \equiv l_2 = l$), then the nonhomogeneous characteristics reduce to a single constant which is contained in the equations as a material constant having the dimension of length. The constant of the moment theory of elasticity (with accuracy up to an unimportant multiplier which is introduced for the sake of convenience) coincides with l, the parameter of denseness of the grid of slip lines. In this case Eq. (4.9) is transformed into the corresponding equation of the moment theory of elasticity: $\partial \mu_{11}/\partial x_2 = \partial \mu_{22}/\partial x_1$ [8].

The equations of equilibrium together with Eqs. (4.1), (4.4) – (4.7), and (4.9) form a closed system relative to

$$\sigma_{11}^0,\ \sigma_{22}^0,\ \sigma_{12}^0,\ \sigma_{21}^0,\ w_1,\ w_2,\ \Omega,\ \Omega_{11}^0,\ \Omega_{22}^0,\ \mu_{11}^0,\ \mu_{22}^0.$$

In Cartesian coordinates the system is transformed into

$$\begin{gathered}
\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + X_1 = 0, \quad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + X_2 = 0,\\
\frac{\sin 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) - \frac{\cos 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) =\\
= \frac{1-2\nu}{2\mu}\frac{\sigma_{11}+\sigma_{22}}{2} + \frac{1}{2\mu}\left[\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta - \frac{\sigma_{12}+\sigma_{21}}{2}\cos 2\theta\right],\\
-\frac{\sin 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\cos 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) =\\
= \frac{1-2\nu}{2\mu}\frac{\sigma_{11}+\sigma_{22}}{2} - \frac{1}{2\mu}\left[\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta - \frac{\sigma_{12}+\sigma_{21}}{2}\cos 2\theta\right],\\
\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) - \Omega + \frac{\sigma_{11}-\sigma_{22}}{4\mu}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{4\mu}\sin 2\theta +\\
+ \frac{1}{f_1 a_1 \varepsilon} S\left[\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{2}\sin 2\theta - \frac{\sigma_{21}-\sigma_{12}}{2}\right],\\
\frac{\cos 2\theta}{2}\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \frac{\sin 2\theta}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + \Omega + \frac{\sigma_{11}-\sigma_{22}}{4\mu}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{4\mu}\sin 2\theta +\\
+ \frac{1}{f_2 a_2 \varepsilon} S\left[\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \frac{\sigma_{12}+\sigma_{21}}{2}\sin 2\theta + \frac{\sigma_{21}-\sigma_{12}}{2}\right],\\
\cos 2\theta\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2}\right) + \sin 2\theta\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) + 2\Lambda\Omega = 0.
\end{gathered} \tag{4.10}$$

(For the sake of brevity of representation, the equations for the moments in Cartesian coordinates are not written out.)

Let $\sigma_{12}^0 < k$ and $\sigma_{21}^0 < k$. Consequently, there is no localized strain and $S(\sigma_{12}^0) \equiv S(\sigma_{21}^0) \equiv 0$, $\Omega = 0$. In this case the angle θ is eliminated from the system (4.10) and the system reduces to the equations of the theory of elasticity. Hence an elastic medium can be defined as a medium in which the condition of continuity of the normal component of displacement (4.1) is fulfilled for all possible directions of λ_1 and λ_2.

We write the system (4.10) in terms of increments:

$$\begin{gathered}
\frac{\partial\Delta\sigma_{11}}{\partial x_1} + \frac{\partial\Delta\sigma_{21}}{\partial x_2} = 0, \quad \frac{\partial\Delta\sigma_{12}}{\partial x_1} + \frac{\partial\Delta\sigma_{22}}{\partial x_2} = 0,\\
\sin 2\theta\left(\frac{\partial\Delta v_1}{\partial x_1} - \frac{\partial\Delta v_2}{\partial x_2}\right) - \cos 2\theta\left(\frac{\partial\Delta v_2}{\partial x_1} + \frac{\partial\Delta v_1}{\partial x_2}\right) = \frac{1}{2\mu}\left[(\Delta\sigma_{11} - \Delta\sigma_{22})\sin 2\theta - (\Delta\sigma_{12} + \Delta\sigma_{21})\cos 2\theta\right];\\
\frac{\partial\Delta v_1}{\partial x_1} + \frac{\partial\Delta v_2}{\partial x_2} = \frac{1-2\nu}{2\mu}(\Delta\sigma_{11} + \Delta\sigma_{22}),\\
\cos 2\theta\left(\frac{\partial\Delta v_1}{\partial x_1} - \frac{\partial\Delta v_2}{\partial x_2}\right) + \sin 2\theta\left(\frac{\partial\Delta v_2}{\partial x_1} + \frac{\partial\Delta v_1}{\partial x_2}\right) = \rho_2(\Delta\sigma_{21} - \Delta\sigma_{12}) + \rho_1\left[(\Delta\sigma_{11} - \Delta\sigma_{22})\cos 2\theta + (\Delta\sigma_{12} + \Delta\sigma_{21})\sin 2\theta\right],\\
2\Delta\Omega = \rho_2\left[(\Delta\sigma_{11} - \Delta\sigma_{22})\cos 2\theta + (\Delta\sigma_{12} + \Delta\sigma_{21})\sin 2\theta\right] + \left(\rho_1 - \frac{1}{2\mu}\right)(\Delta\sigma_{21} - \Delta\sigma_{12}),\\
\cos 2\theta\left(\frac{\partial\Delta v_1}{\partial x_1} - \frac{\partial\Delta v_2}{\partial x_2}\right) + \sin 2\theta\left(\frac{\partial\Delta v_2}{\partial x_1} + \frac{\partial\Delta v_1}{\partial x_2}\right) + 2\Lambda\Delta\Omega = 0,
\end{gathered} \tag{4.11}$$

where

$$2\rho_1 = \frac{S'(\sigma_{12}^0)}{f_1 a_1 \varepsilon} + \frac{S'(\sigma_{21}^0)}{f_2 a_2 \varepsilon} + \frac{1}{\mu}; \quad 2\rho_2 = \frac{S'(\sigma_{21}^0)}{f_2 a_2 \varepsilon} - \frac{S'(\sigma_{12}^0)}{f_1 a_1 \varepsilon}.$$

We consider the problem of the coefficients b_1 and b_2 which figure in Eqs. (3.7), i.e., the problem of functioning of slip lines from different families. Let (λ_1, λ_2) be the coordinates of a certain initial element, while $(\lambda_1 + \Delta\lambda_1, \lambda_2)$, $(\lambda_1, \lambda_2 + \Delta\lambda_2)$, and $(\lambda_1 + \Delta\lambda_1, \lambda_2 + \Delta\lambda_2)$ are the coordinates of the elements bordering on the initial elements. The compatibility conditions of strain of the elements (λ_1, λ_2), $(\lambda_1 + \Delta\lambda_1, \lambda_2)$, and $(\lambda_1, \lambda_2 + \Delta\lambda_2)$ lead to Eqs. (3.1) or (4.1). At the same time, the condition of compatibility of strain of the elements (λ_1, λ_2) and $(\lambda_1 + \Delta\lambda_1, \lambda_2 + \Delta\lambda_2)$ must be satisfied, which leads to the fact that at any specified moment of time t, $b_1(t)b_2(t) = 0$, i.e., in the medium being considered only alternating functioning of slip lines from different families (turbulent plastic flow [9]) is possible. Therefore, the increments of all sought variables consist of two parts: one part (with the index "−") satisfies Eqs. (4.11) (in which we have put $b_1 = 0$), while the other (with the index "+") satisfies the same equations but with $b_2 = 0$. Since both systems are linear, by adding the corresponding equations relative to increments with the indices "+" and "−" we obtain as the sum the same system (4.11), in which $\Lambda = [(\Delta-\Omega)-(\Delta+\Omega)][(\Delta-\Omega)+(\Delta+\Omega)]$. Here it is necessary to take into account the fact that $S'(\sigma_{12}^0)$, $S'(\sigma_{21}^0)$, and, consequently, ρ_1, ρ_2 depend on the sign of the increments $\Delta\sigma_{12}^0$ and $\Delta\sigma_{21}^0$.

If the quantity Λ is known (for example, from the symmetry conditions $\Lambda = 0$ or from the boundary conditions, as in the problem of the die, $\Lambda = 1$, and so forth), then we can at once solve the system for the increments with the index "+" or "−". Here the question arises as to during which time intervals the boundary conditions are satisfied as a result of slip along one family and as a result of slip along the other family. In each problem this question is resolved from additional considerations, taking into account the actual loading conditions of the material.

We consider the problem concerned with the type of the system (4.11). From the last five equations we express $\Delta\sigma_{km}$ in terms of Δv_1 and Δv_2. Then, substituting the expressions for $\Delta\sigma_{km}$ into the equations of equilibrium, we obtain two quasilinear equations of the second order relative to Δv_1 and Δv_2. By $\varkappa$ we denote the tangent of the angle of inclination of the characteristic to the Ox_1 axis in the local coordinates $\theta = \pi/4$. Then

$$\varkappa = \pm \sqrt{\frac{-(1-\nu\xi_1) \pm \sqrt{(1-\nu\xi_1)^2 - (1-\nu)^2(\xi_1^2 - \xi_2^2)}}{(1-\nu)(\xi_1 + \xi_2)}}, \qquad (4.12)$$

where

$$\xi_1 = \frac{1}{2\mu} \frac{\rho_1 + \rho_2/\Lambda - 1/4\mu}{\rho_1^2 - \rho_2^2 - \rho_1/4\mu};$$

$$\xi_2 = \frac{1}{2\mu} \frac{\rho_1/\Lambda + \rho_2}{\rho_1^2 - \rho_2^2 - \rho_1/4\mu}.$$

The type of the system (4.11) is determined by the signs of the expressions under the square root sign in relation (4.12).

The expression (4.12) shows that the type of the system (4.11) depends on the derivatives of the functions S. We consider the mechanical significance of this relation when $\sigma_{12}^0 \equiv \sigma_{21}^0$. Here we have the conditions

$$\xi_1 = \frac{1}{2\mu}\frac{1}{\rho_1}, \quad \xi_2 = 0, \quad \Lambda = -\frac{\rho_1}{\rho_2}. \qquad (4.13)$$

The relation (4.12) [with (4.13) taken into account] shows that the system (4.11) is of elliptic type in the case where the material hardens ($\xi_1 > 0$), and is of hyperbolic type in the case of an ideal flow of the material ($\xi_1 = 0$). If the material softens and its softening is not too intense $[-1/(1-2\nu) \leqslant \xi_1 < 0]$, then the system also is of hyperbolic type. If, however, the softening of the material becomes sufficiently intense $[-\infty < \xi_1 < -1/(1-2\nu)]$, then the type of the system changes to elliptic.

This result, which is paradoxical at first glance, can be clarified using the following example. Let the strip $A_1A_2A_3A_4$ be stretched in the x_1 direction (Fig. 5). We assume that on the boundary and within the region of deformation $\sigma_{12} = \sigma_{21} = \sigma_{22} = 0$ and $\theta = 0$. From Eqs. (4.11) we can find the connection of the increment in the length of the strip with the tensile stresses:

$$\frac{\Delta v_1}{\Delta\sigma_{11}} = \left[\frac{1}{\xi_1} + (1-2\nu)\right]\frac{x_1}{4\mu}. \qquad (4.14)$$

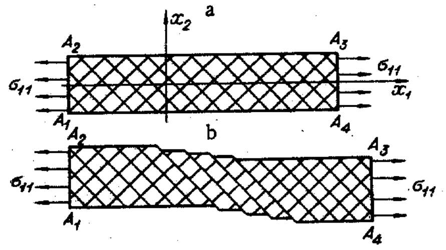


Fig. 5

If the material hardens, then the signs of Δv_1 and $\Delta\sigma_{11}$ are the same, i.e., an increase or decrease in the tensile force leads to an increase or decrease in the length of the strip. Now let the increase in the localized shear on the slip plane lead to a decrease in the shear stresses: the material softens. On the one hand, the increase in the localized shear leads to an increase in the length of the strip. On the other hand, the decrease in the shear stresses leads to elastic unloading of the elements, which gives rise to a decrease in the length of the strip.

For $\xi_1 > -1/(1-2\nu)$ the first factor prevails over the second, and for a decreasing external load an overall elongation of the strip takes place: The system (4.11) is of hyperbolic type. If, however, $\xi_1 < -1/(1-2\nu)$, then the second factor will be the dominant one: For a decreasing tensile force and increasing localized shear the overall length of the strip will decrease as a result of the elastic compression of the elements, i.e., in outward appearance the material behaves as a hardening material, and the type of the system is changed to elliptical. If $\xi_1 = -1/(1-2\nu)$, then the tensile force decreases for a constant length of the strip. Under these conditions the system is energetically isolated and a transition of the stored energy into work performed on the slip planes takes place within it.

We consider yet another circumstance. Let the tensile stress σ_{11} increase monotonically from zero and then, if there is a falling branch, let it decrease. Then on the rising branch the function $S(\sigma_{11}/2)$ is single-valued and, consequently, $\rho_2 = 0$. From Eqs. (4.11) it follows that in this case $\Delta\Omega = 0$, i.e., the localized shears on both families of slip lines are the same and the lines function symmetrically (see Fig. 5a). If the material begins to soften, then the function $S(\sigma_{11}/2)$ becomes non-single-valued, and the possibility of the other mode of deformation appears, when $\rho_2 \neq 0$ and $\Delta\Omega \neq 0$ (see Fig. 5b). This signifies that the decrease of the shear stresses on one of the planes takes place as a result of an increase in the shear, while on the other plate it occurs as a result of unloading. If we take into account the alternate character of the functioning of the slip lines, then we can draw the conclusion that on the falling branch the second mode of deformation is being realized. Without dwelling on problems of the general formulation of boundary-value problems and of existence, uniqueness, and stability of the solutions (4.10), we note that we can expect nonuniqueness of solution on the falling branch. At the same time, some of the solutions will be unstable. A stability analysis provides a natural criterion of choice of the solution.

It can be shown that in the case of free surfaces A_1A_4 and A_2A_3 the solution (4.14) is unstable and nonunique. Since the role of this solution is illustrative, we assume that kinematic constraints, which result in disturbances leading to instability, are specified on the boundaries A_1A_4 and A_2A_3. The solution (4.14) points to yet another peculiarity inherent to deformation of elastoplastic hardening materials. From (4.14) it follows that the "tensile force–elongation of strip" diagram can have the form depicted in Fig. 6. We assume that the loading takes place with a controlled monotonically increasing elongation. Then for an elongation equal to v_1^* there occurs in the material a noncontrolled liberation of a part of the elastic potential energy, which corresponds to the transition from the point A_1 to A_2 and then to A_3. Here, in the transition from A_1 to A_2, the liberated potential energy is entirely dissipated on the slip lines, while in the transition from A_2 to A_3 only a part of the energy is dissipated; the rest is transformed into the kinetic energy of the elements. If such a "discharge" does not destroy the specimen, then its subsequent deformation will take place along the branch A_3A_4.

We consider the axisymmetric solution of the problem of the stress–strain state of the material around a circular hole. We introduce the system of polar coordinates (r, β) and assume that $\sigma_{\beta r} = \sigma_{r\beta} = 0$ on the boundary and inside the region of deformation. Hence $\theta \equiv \beta$ and the slip lines will be logarithmic spirals. Let the elements be regular and their dimensions not depend on β: $f_1(\lambda_1) = f_2(\lambda_2) = 1$, $l_1 = l_2 = \varepsilon r$. We also assume that mass forces are absent. Then the system (4.10) is transformed into

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\beta\beta}}{r}=0,\quad \frac{\partial v_r}{\partial r}+\frac{v_r}{r}=\frac{1-2\nu}{2\mu}(\sigma_{rr}+\sigma_{\beta\beta}),$$
$$\frac{\partial v_r}{\partial r}-\frac{v_r}{r}=\frac{2}{\varepsilon r}G\left(\frac{\sigma_{rr}-\sigma_{\beta\beta}}{2}\right)+\frac{\sigma_{rr}-\sigma_{\beta\beta}}{2\mu},$$

where

$$2G=S(\sigma_{12}^0)+S(\sigma_{21}^0);\quad \sigma_{12}^0=\sigma_{21}^0=\frac{\sigma_{rr}-\sigma_{\beta\beta}}{2}.$$

Expressing the half-difference of the principal stresses in terms of the radial displacement from the first two equations and then using the last equation, we obtain the general solution in the form

$$v_r=2r\int\frac{w}{r}dr+C_1r,\quad \sigma_{rr}=\tau+\frac{\mu}{1-2\nu}\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right),$$
$$\sigma_{\beta\beta}=-\tau+\frac{\mu}{1-2\nu}\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}\right), \tag{4.15}$$
$$\omega=\frac{1}{2\varepsilon r}\left[S(\sigma_{21}^0)-S(\sigma_{12}^0)\right],$$

where $w=\frac{1-2\nu}{2\mu}\left(\frac{C_2}{r^2}-\tau\right)$, and τ satisfies the final relation

$$2\mu G(\tau)+2\varepsilon r(1-\nu)\tau=\frac{\varepsilon C_2(1-2\nu)}{r} \tag{4.16}$$

(C_1 and C_2 are the integration constants). On the rising branch the function S is single-valued and $G\equiv S$, $\omega=0$, i.e., on the rising branch the slip lines from the different families function identically. On the falling branch the function S is non-single-valued, $\omega\neq0$, and localized shear (slip) continues only along one of the families of slip lines.

If $G(\tau)$ is indeterminate for a certain $\tau=k$ (ideal flow), then Eq. (4.16) is replaced by the equation $\tau=k$ and (4.15) is transformed into the usual solution of the theory of ideal plasticity. For $G(\tau)\equiv0$ (no local strain, elasticity), Eq. (4.15) is transformed into the solution of the theory of elasticity.

In the presence of a falling part of the $G(\tau)$ diagram the solution (4.16) can be nonunique, pointing to the possibility of a noncontrolled liberation of the stored elastic energy in a certain region of the material being deformed.

Thus, we have considered the deformation of a material divided into elements by a discrete grid of slip lines. The distances between the slip lines were assumed to be sufficiently small, so that the transition into differential equations did not incur large errors. For a numerical solution of the problems an inverse transition into a "discrete" model is required. It is obvious that in the case of formulation of problems for numerical calculation there is no need to make two transitions: The problem can be posed at once for a discrete grid of slip lines. Here two circumstances can be used: for an elastic body we can take any convenient grid of slip lines of arbitrary orientation and denseness, while for an ideally plastic body we can take a grid of any denseness.

The first circumstance allows us to consider elastoplastic problems, when the distance between the actual slip lines (on which the slip is nonzero) is comparable with the characteristic dimension of the body being deformed. Above, the determining relations for the stresses–strains of the elements and the stresses–slips between the elements were written out separately. The first relations were assumed to be purely elastic or rigid, while the second relations were assumed to be purely plastic. These restrictions are not of a

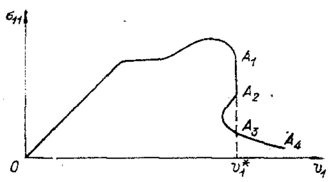


Fig. 6

major character and admit a generalization of relations both of the first and of the second types. In particular, we can take into account creep, nonorthogonality of the slip lines, dilatational effects [10], and effects of internal friction which have importance for soils and rocks.

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FORMING OF FIBROUS LIGHTGUIDES WITH A SMALL AZIMUTHAL ASYMMETRY OF THE BILLET

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UDC 532.5:535.8:666.189.2

One possible type of fibrous lightguide is a transparent microcapillary. Small losses with the propagation of light along a lightguide are possible if its transverse cross section is sufficiently close to a concentric round ring and is constant over the length of a fiber. From a physical point of view, the process of the forming of a lightguide can be represented as the flow of an incompressible Newtonian liquid with a variable viscosity (some polymers are not Newtonian liquids and are therefore not discussed here).

Article [1] discusses the pulling of a microcapillary from a billet, i.e., a solid hollow cylinder of given dimensions. The billet and all the external conditions under which the pulling was done were assumed to be axisymmetric, as a result of which the microcapillary pulled was also axisymmetric with a round cross section. In [1] equations for the form of the jet (the transition from the billet to the microcapillary) were obtained and the dependence of the dimensions of the microcapillary on the parameters of the process was found. We discuss below the pulling of a microcapillary from a billet, taking account of the small real nonaxisymmetric character of the latter; the degree of nonaxisymmetry of the microcapillary is found and its dependence on the parameters of the process is investigated.

§1. In all aspects, except for the assumption of the nonaxisymmetry of the process, the statement of the problem is the same as in [1]: the temperature distribution is assumed to be given; in all cross sections, the thickness of the wall of the billet and the jet is assumed to be small in comparison with its radius; by virtue of the thinness of the wall, the temperature is assumed to be identical at all points of the transverse cross section of the jet and to depend only on the longitudinal coordinate z; the viscosity is a known function of the tem-

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 175-182, May-June, 1977. Original article submitted May 28, 1976.